



MINISTRY OF HIGHER EDUCATION THE HIGHER INSTITUTE FOR ENGINEERING & TECHNOLOGY IN NEW DAMIETTA		
1 st SEMESTER 2017-2018		
COURSE TITEL: Math 1	DATE: 11 / 11 / 2017	DAY: SATURDAY
COURSE CODE: MTH 101	TIME ALLOWED: 1½ hours	
Total Mark: 20	Model Answer	

Question 1 (7 marks)

- a) For the two sets, $A=\{1,2,3\}$ and $B=\{3,4,5,9\}$; (2 marks)
 1- State whether each of the following defines a function from A to B, or not. Explain your answer.

Solution

i) $f=\{(1,5), (2,8),(3,9)\}$

is not a function ----- where $f(2)=8 \notin B$

ii) $g(a)=a +2, a \in A$

is a function ----- where $g(1)=3 \in B, g(2)=4 \in B$ and $g(3)=5 \in B$

- 2- Get the inverse of both (f and g). Are their inverse functions?

$f^{-1}=\{(5,1), (8,2),(9,3)\}$ is not a function where $8 \notin B$

$g^{-1}=\{(3,1), (4,2),(5,3)\}$ is not a function where $g^{-1}(9) \notin A$

- b) Suppose that: f_1, f_2, f_3 and f_4 are four functions where, (2 marks)

$dom(f_1) =[-1,1]$, $dom(f_2) = (-\infty,0]$, $f_3(x)=x^3+1$ and $f_4(x)=5x+3$;

Find the following:

$dom(3f_1f_2)$, $dom(f_1/f_3)$, $(f_3 \circ f_4)(2)$, $(f_4 \circ f_3)(x)$.

Solution

1) $dom(3f_1f_2) = dom(f_1) \cap dom(f_2) = [-1,0]$

2) $dom(f_1/f_3) = dom(f_1) \cap dom(f_3)$, $f_3 \neq 0$ and $dom(f_3) = R$

$dom(f_1/f_3) = (-1,1]$

3) $(f_3 \circ f_4)(2) = (f_3(f_4(2))) = f_3(5(2)+3) = f_3(13) = (13)^3+1 = 2198$

4) $(f_4 \circ f_3)(x) = (f_4(f_3(x))) = f_4(x^3+1) = 5(x^3+1) + 3 = 5x^3 + 8$

- c) Given that f and g are two functions defined as follows: (3 marks)

$f:R \rightarrow R$ where $f(x)=3x^2+4$ and $g:R \rightarrow R$ where $g(x)=7x+5$,

Determine whether:

Solution

- i) $g(x)$ is bijective or not?

1) let $g(x_1)=g(x_2)$

$7x_1+5=7x_2+5 \rightarrow 7x_1=7x_2 \rightarrow x_1=x_2 \rightarrow g(x_1)$ is injective (one-to-one).

- 2) The range of $g(x)$ is R . That is, $g:R \rightarrow R$ is surjective (onto).

Then $g(x)$ is bijective

- ii) $f(x)$ is even, odd or neither?

$f(-x)=3(-x)^2+4=3x^2+4=f(x)$

Then $f(x)$ is even

Question 2 (6 marks)

a) Consider the four functions f , g , h and s , be such that: (2 marks)

$$\lim_{x \rightarrow 1} f(x) = 2, \lim_{x \rightarrow 1} g(x) = -3, h(x) = \sqrt[3]{x^2 + 5x + 2}$$

and that $f(x) \leq s(x) \leq h(x)$, $\forall x$ in their domain;

Evaluate the following:

Solution

$$\lim_{x \rightarrow 1} h(x) = \sqrt[3]{1+5+2} = 2 \quad \text{and} \quad \lim_{x \rightarrow 1} s(x) = 2 \text{ using the squeeze theory}$$

$$\text{i)} \lim_{x \rightarrow 1} \left[\frac{f(x) \cdot g(x)}{s(x)} \right] - h(x) = \left(\frac{2(-3)}{2} \right) - 2 = -5$$

$$\text{ii)} \lim_{x \rightarrow 1} \left(\frac{s(x)}{4} + g(x) + 7 \right) = \frac{2}{4} + (-3) + 7 = \frac{9}{2}$$

b) Study the continuity of each the following functions at $x=2$: (2 marks)

Solution

$$\text{i)} f(x) = |x-2| = \begin{cases} x-2 & x \geq 2 \\ 2-x & x < 2 \end{cases}$$

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2} (x-2) = 0 \quad \text{and} \quad \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2} (2-x) = 0$$

$$\lim_{x \rightarrow 2} f(x) = 0 \quad \text{the limit exists}$$

$$\text{and} \quad f(2) = 2-2 = 0 \quad \text{that is} \quad \lim_{x \rightarrow 2} f(x) = f(2)$$

then $f(x)$ is continuous at $x=2$.

$$\text{ii)} g(x) = \begin{cases} \frac{x^2-x-2}{x^2-3x+2} & x > 2 \\ 3x^2-x-5 & x \leq 2 \end{cases}$$

$$\lim_{x \rightarrow 2^+} g(x) = \lim_{x \rightarrow 2} \left(\frac{x^2-x-2}{x^2-3x+2} \right) = \lim_{x \rightarrow 2} \frac{(x-2)(x+1)}{(x-2)(x-1)} = \lim_{x \rightarrow 2} \frac{x+1}{x-1} = 3$$

$$\lim_{x \rightarrow 2^-} g(x) = \lim_{x \rightarrow 2} (3x^2 - x - 5) = 12 - 2 - 5 = 5$$

Then the limit $\lim_{x \rightarrow 2} g(x) = 0$ does not exist

Hence, $g(x)=0$ is not continuous at $x=2$.

c) Choose the suitable answer (2 marks)

Solution

1- If $\lim_{x \rightarrow \infty} \frac{x^k - x - 2}{x^4 - 3x + 2} = 0$, then

$$\text{i)} k = 3$$

2- For the function $\frac{x^4 - x^3 - 2x + 1}{x^4 + x^2 - 3x + 2}$, the line $y=1$, is said to be its

$$\text{ii) vertical asymptote}$$

3- As x gets small, the expression $\left(\frac{\sin x}{x}\right)$

ii) gets closer and closer to 1.

4- As x gets small, the expression $\left(\sin\frac{1}{x}\right)$

iv) isn't getting closer to any particular number.

Question 3 (7 marks)

a) Use the definition to deduce the derivative of the function (e^x) . (1 mark)

Solution

$$\frac{d(e^x)}{dx} = \lim_{\Delta x \rightarrow 0} \frac{e^{x+\Delta x} - e^x}{\Delta x} = e^x \lim_{\Delta x \rightarrow 0} \frac{e^{\Delta x} - 1}{\Delta x} = e^x$$

b) Find $\frac{dy}{dx}$, for each of the following functions: (4 marks)

Solution

i) $x = a(1 + \sin \theta), \quad y = a(1 - \cos \theta)$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a(\sin \theta)}{a(\cos \theta)} = \frac{x-a}{a-y}$$

ii) $y = \pi^x x^{3\pi} - \cosh(\ln(3x^2)) + \sqrt{x + \sqrt{x}}$

$$\frac{dy}{dx} = [\pi^x \cdot 3\pi \cdot x^{3\pi-1} + \pi^x \cdot \ln \pi \cdot x^{3\pi}] - [\sinh(\ln(3x^2)) \frac{6x}{3x^2}] + \frac{1 + \frac{1}{2\sqrt{x}}}{2\sqrt{x + \sqrt{x}}}$$

iii) $y = \left(\frac{(x-1)(x^2-2)}{(x^3-5)(x+1)} \right)^{\frac{1}{2}}$

$$\ln(y) = \frac{1}{2} (\ln(x-1) + \ln(x^2-2) - \ln(x^3-5) - \ln(x+1))$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{2} \left(\frac{1}{x-1} + \frac{2x}{x^2-2} - \frac{3x^2}{x^3-5} - \frac{1}{x+1} \right)$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{1}{x-1} + \frac{2x}{x^2-2} - \frac{3x^2}{x^3-5} - \frac{1}{x+1} \right) \left(\frac{(x-1)(x^2-2)}{(x^3-5)(x+1)} \right)^{\frac{1}{2}}$$

iv) $\cos(x-y) = x \sin y$

$$-\sin(x-y)\left(1 - \frac{dy}{dx}\right) = x \cos y \frac{dy}{dx} + \sin y$$

$$-\sin(x-y) + \sin(x-y) \frac{dy}{dx} = x \cos y \frac{dy}{dx} + \sin y$$

$$(\sin(x-y) - x \cos y) \frac{dy}{dx} = \sin y + \sin(x-y)$$

$$\frac{dy}{dx} = \frac{\sin y + \sin(x-y)}{\sin(x-y) - x \cos y}$$

c) Prove that: $\tanh^{-1}(x) = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right|$ (1 mark)

Solution

$$\text{Let } y = \tanh^{-1}(x) \Rightarrow \tanh(y) = x$$

$$\frac{e^y - e^{-y}}{e^y + e^{-y}} = x \Rightarrow e^y - e^{-y} = x(e^y + e^{-y})$$

$$e^{2y} - 1 = x(e^{2y} + 1) \Rightarrow e^{2y}(1 - x) = 1 + x$$

$$e^{2y} = \left| \frac{1+x}{1-x} \right|, \quad \text{where } e^{2y} > 0$$

$$2y = \ln \left| \frac{1+x}{1-x} \right| \Rightarrow y = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right|$$

d) If $\sinh(x) = \frac{4}{3}$, compute the values: $\cosh(x)$ and $\tanh(x)$. (1 mark)

Solution

$$\cosh^2(x) - \sinh^2(x) = 1 \Rightarrow \cosh(x) = \sqrt{1 + \sinh^2(x)} = \frac{5}{3}, \text{ where } \cosh(x) > 0$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{\frac{4}{3}}{\frac{5}{3}} = \frac{4}{5}$$