



MINISTRY OF HIGHER EDUCATION

THE HIGHER INSTITUTE FOR ENGINEERING & TECHNOLOGY IN NEW DAMIETTA

Department: Basic Science

1st SEMESTER 2017-2018

Final Exam (Model Answer)

COURSE TITEL: Math 1

DATE: 13 / 1 / 2018 DAY: SATURDAY

COURSE CODE: MTH 101

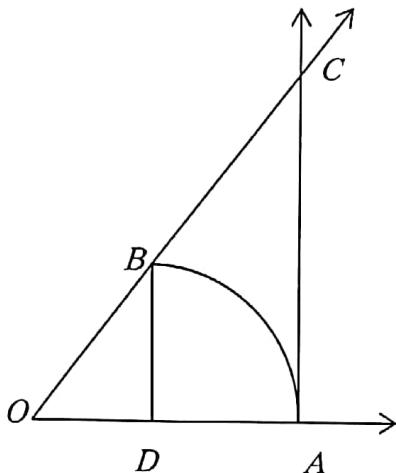
TIME ALLOWED: 3 hours

Total Mark: 60

No. of exam pages: 2 pages (1 sheet)

Question 1 (20 marks)

a) Prove that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$. (4 marks)



From the figure

$$\begin{aligned}|OB| &= |OA| = 1 \\ |OD| &= \cos x \\ |BD| &= \sin x \\ |AC| &= \tan x \\ \widehat{AB} &= x\end{aligned}$$

From the figure

$$|BD| \leq \widehat{AB} \leq |AC|$$

$$\sin x \leq x \leq \tan x$$

$$1 \leq \frac{x}{\sin x} \leq \frac{1}{\cos x}$$

$$\lim_{x \rightarrow 0} 1 \leq \lim_{x \rightarrow 0} \frac{x}{\sin x} \leq \lim_{x \rightarrow 0} \frac{1}{\cos x}$$

$$1 \leq \lim_{x \rightarrow 0} \frac{x}{\sin x} \leq 1$$

Then $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

b) Obtain the following limits:

i) $\lim_{x \rightarrow \infty} \frac{x-3}{x^2 + x - 6}$ (2 marks)

$$\lim_{x \rightarrow \infty} \frac{x-3}{x^2 + x - 6} = \lim_{x \rightarrow \infty} \frac{1}{2x+1} = 0$$

ii) $\lim_{x \rightarrow 0} (\cot x)^{\sin x}$ (3 marks)

$$y = (\cot x)^{\sin x} \Rightarrow \ln y = \sin x \ln(\cot x)$$

$$\lim_{x \rightarrow 0} \ln y = \lim_{x \rightarrow 0} \sin x \ln(\cot x) = \lim_{x \rightarrow 0} \frac{\ln(\cot x)}{\csc x} = \frac{0}{0}$$

$$\lim_{x \rightarrow 0} \ln y = \lim_{x \rightarrow 0} \frac{-\csc^2 x / \cot x}{-\csc x \cot x} = 0$$

$$\ln \lim_{x \rightarrow 0} y = 0 \Rightarrow \lim_{x \rightarrow 0} y = e^0 = 1$$

c) Find the inverse of the function $f(x) = \frac{x+4}{2x-5}$. (2 marks)

$$\text{Let } x = \frac{y+4}{2y-5}$$

$$x = \frac{y+4}{2y-5}$$

$$x(2y-5) = y+4$$

$$(2x-1)y = 5x+4$$

$$y = \frac{5x+4}{2x-1}$$

$$\text{Then } f^{-1}(x) = \frac{4+5x}{2x-1}$$

d) Determine where the function $f(x) = \frac{4x+10}{x^2-2x-15}$ is continuous. (3 marks)

the function is an algebraic fraction, where both the denominator and numerator functions are polynomials.

Since the polynomials are continuous on \mathbb{R} , the function $f(x) = \frac{4x+10}{(x-5)(x+3)}$ is continuous on $\mathbb{R} - \{-3, 5\}$. Where $\{-3, 5\}$ are the zeros of the denominator function.

e) Considering $x^2 + y^2 - 2x - 6y + 5 = 0$, evaluate $\frac{d^2y}{dx^2}$. (3 marks)

$$2x + 2y \frac{dy}{dx} - 2 - 6 \frac{dy}{dx} = 0$$

$$\therefore (2y-6) \frac{dy}{dx} = 2-2x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2-2x}{2y-6} = \frac{1-x}{y-3}$$

$$\text{Then } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{1-x}{y-3} \right) = \frac{(y-3)(-1)-(1-x)\frac{dy}{dx}}{(y-3)^2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{(3-y)-(1-x)\frac{dy}{dx}}{(y-3)^2}$$

$$\frac{d^2y}{dx^2} = \frac{(3-y)-(1-x)\frac{(1-x)}{(y-3)}}{(y-3)^2}$$

f) If $y = \tan^{-1} \left\{ \frac{1+\tan x}{1-\tan x} \right\}$, show that $\frac{dy}{dx} = 1$. (3 marks)

$$\frac{dy}{dx} = \frac{1}{1 + \left(\frac{1+tanx}{1-tanx}\right)^2} \frac{(1-tanx)(secx)^2 + (1+tanx)(secx)^2}{(1-tanx)^2}$$

$$\frac{dy}{dx} = \frac{(1-tanx)^2}{(1-tanx)^2 + (1+tanx)^2} \frac{(secx)^2[(1-tanx) + (1+tanx)]}{(1-tanx)^2}$$

$$\frac{dy}{dx} = \frac{2(secx)^2}{1 - 2tanx + (tanx+)^2 + 1 + 2tanx + (tanx+)^2}$$

$$\frac{dy}{dx} = \frac{2(secx)^2}{2 + 2(tanx+)^2} = \frac{2(secx)^2}{2(secx)^2} = 1$$

Question 2 (20 marks)

- a) Deduce the nth derivative for the function $y = \sin(ax+b)$, where a, b are real constants.
(3 marks)

$$y' = a \cos(ax+b) = a \sin\left(ax+b+\frac{\pi}{2}\right)$$

$$y'' = a^2 \cos\left(ax+b+\frac{\pi}{2}\right) = a^2 \sin\left(ax+b+2\frac{\pi}{2}\right)$$

$$y''' = a^3 \cos\left(ax+b+2\frac{\pi}{2}\right) = a^3 \sin\left(ax+b+3\frac{\pi}{2}\right)$$

$$y^{(n)} = a^n \sin\left(ax+b+n\frac{\pi}{2}\right)$$

- b) If $y = 5 \cos(\ln x) + 3 \sin(\ln x)$, show that:

$$\text{i) } x^2 y_2 + xy_1 + y = 0 \quad \underline{(2 \text{ marks})}$$

$$\text{ii) } x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0 \quad \underline{(3 \text{ marks})}$$

$$\text{i) } y_1 = \frac{-5}{x} \sin(\ln x) + \frac{3}{x} \cos(\ln x)$$

$$xy_1 = -5 \sin(\ln x) + 3 \cos(\ln x)$$

$$xy_2 + y_1 = \frac{-5}{x} \cos(\ln x) - \frac{3}{x} \sin(\ln x)$$

$$x^2 y_2 + xy_1 = -(5 \cos(\ln x) + 3 \sin(\ln x)) = -y$$

$$x^2 y_2 + xy_1 + y = 0$$

$$\text{ii) } [x^2 y_2]_n + [xy_1]_n + [y]_n = 0$$

using Leibnitz theory

$$[x^2 y_{n+2} + 2nxy_{n+1} + n(n-1)y_n] + [xy_{n+1} + ny_n] + y_n = 0$$

$$x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$$

- c) Get the critical points of the function $f(x) = 3x^5 - 5x^3 + 3$; hence, use the second derivative test to classify these critical points. (4 marks)

$$f'(x) = 15x^4 - 15x^2 = 15x^2(x-1)(x+1)$$

$$\text{When } f'(x) = 0$$

the critical points are $x = 0, x = -1, x = 1$

$$f''(x) = 60x^3 - 30x$$

$f''(1) = 30$ then $x = 1$ is a relative minimum

$f''(-1) = -30$ then $x = -1$ is a relative maximum

$f''(0) = 0$ then the test fails to determine whether $x = 0$ is a relative minimum or maximum

d) **Verify** Rolle's theorem for the function $f(x) = x^2 - 3x$, in the interval $[0,3]$. (4 marks)

To apply Rolle's theorem for the function, we check

1- The function a polynomial, that is continuous and differentiable on .
hence, $f(x)$ is continuous in the interval $[0,3]$ and differentiable in the interval $(0,3)$.

2- , $f(0) = 0$ and $f(3) = 9 - 9 = 0$

3- Then there exists at least one point $c \in (0,3)$ where $f'(c) = 0$

$$f'(x) = 2x - 3$$

$$f'(c) = 2c - 3 = 0$$

then $c = \frac{3}{2} \in (0,3)$ which verify the theorem

e) **Find** Taylor's expansion for the function $f(x) = e^{\frac{x}{2}}$, about the point $x_o = 2$. (4 marks)

$$f(x) \approx f(x_o) + \frac{f'(x_o)}{1!}(x - x_o) + \frac{f''(x_o)}{2!}(x - x_o)^2 + \dots + \frac{f^{(n)}(x_o)}{n!}(x - x_o)^n + \dots$$

$$f(x) = e^{\frac{x}{2}} \quad f(2) = e$$

$$f'(x) = \frac{1}{2}e^{\frac{x}{2}} \quad f'(2) = \frac{1}{2}e$$

$$f''(x) = \frac{1}{4}e^{\frac{x}{2}} \quad f''(2) = \frac{1}{4}e$$

$$e^{\frac{x}{2}} \approx f(2) + \frac{f'(2)}{1!}(x - 2) + \frac{f''(2)}{2!}(x - 2)^2 + \dots + \frac{f^{(n)}(2)}{n!}(x - 2)^n + \dots$$

$$e^{\frac{x}{2}} \approx e \left[1 + \frac{1}{2}(x - 2) + \frac{1}{8}(x - 2)^2 + \dots + \frac{1}{(2)^n n!}(x - 2)^n + \dots \right]$$

Question 3 (20 marks)

- a) Use the mathematical induction procedure to prove that: $1^3 + 2^3 + 3^3 + \dots + n^3 = \left[\frac{n(n+1)}{2} \right]^2$ is true for all integers $n \geq 1$. (3 marks)

Step 1: at $n = 1$

$$RS = 1^3 = 1$$

$$LS = \left(\frac{1(2)}{2} \right)^2 = 1$$

Then the relation is true for $n = 1$

Step 2: at $n = k$

$$\text{Let } 1^3 + 2^3 + \dots + k^3 = \left(\frac{k(k+1)}{2} \right)^2$$

Step 3: at $n = k + 1$

$$\begin{aligned} RS &= 1^3 + 2^3 + \dots + k^3 + (k+1)^3 = \left(\frac{k(k+1)}{2} \right)^2 + (k+1)^3 \\ &= (k+1)^2 \left[\left(\frac{k}{2} \right)^2 + (k+1) \right] = \frac{(k+1)^2}{2^2} [k^2 + 4(k+1)] = \frac{(k+1)^2}{2^2} [(k+2)^2] \\ &= \left[\frac{(k+1)(k+2)^2}{2} \right] = LS \end{aligned}$$

Then the relation is true for $n = k + 1$

Then the relation is true for all $n \geq 1$.

- b) Express $\frac{18x^2 + 3x + 6}{(3x+1)^3}$ in partial fractions. (3 Marks)

$$\frac{18x^2 + 3x + 6}{(3x+1)^3} = \frac{A}{3x+1} + \frac{B}{(3x+1)^2} + \frac{C}{(3x+1)^3}$$

$$\frac{18x^2 + 3x + 6}{(3x+1)^3} = \frac{2}{(3x+1)} - \frac{3}{(3x+1)^2} + \frac{7}{(3x+1)^3}$$

- c) Calculate the value of c that makes the matrix $A = \begin{pmatrix} 3 & 2 & 5 \\ 4 & 7 & 9 \\ 1 & 5 & c \end{pmatrix}$ singular. (3 Marks)

The matrix A is singular when $|A| = 0$

$$|A| = \begin{vmatrix} 3 & 2 & 5 \\ 4 & 7 & 9 \\ 1 & 5 & c \end{vmatrix} = 0 \text{ then } c = 4$$

- d) Evaluate the inverse of the matrix $B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$. (3 Marks)

construct the matrix

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\begin{array}{l} R_2 - 2R_1 \\ R_3 - R_1 \end{array}} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array} \right]$$

$$\xrightarrow{\begin{array}{l} R_1 - 2R_2 \\ R_3 + 2R_2 \end{array}} \left[\begin{array}{ccc|ccc} 1 & 0 & 9 & 5 & -2 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array} \right] \xrightarrow{-R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 9 & 5 & -2 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array} \right]$$

$$\xrightarrow{\begin{array}{l} R_1 - 9R_3 \\ R_2 + 3R_3 \end{array}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array} \right]$$

Then $B^{-1} = \begin{bmatrix} 40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$

e) For the two matrices $G = \begin{pmatrix} 4 & 3 \\ 2 & 7 \\ 6 & 1 \end{pmatrix}$ and $H = \begin{pmatrix} 5 & 9 & 2 \\ 4 & 0 & 8 \end{pmatrix}$, compute the following:

i) $2G + H^t$ ii) GH (4 Marks)

i) $2G + H^t = \begin{pmatrix} 13 & 10 \\ 13 & 14 \\ 14 & 10 \end{pmatrix}$

ii) $GH = \begin{bmatrix} 32 & 36 & 32 \\ 38 & 18 & 60 \\ 34 & 54 & 20 \end{bmatrix}$

f) Use the matrices to determine whether the following system consistent or inconsistent:

$$\begin{aligned} x + 2y - z &= 1 \\ 2x - y &= 3 \\ 3x + y - z &= 5 \end{aligned} \quad (4 \text{ Marks})$$

construct the matrix

$$= \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & -1 & 0 & 3 \\ 3 & 1 & -1 & 5 \end{array} \right] \xrightarrow{\begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -5 & 2 & -3 \\ 0 & -5 & 2 & 2 \end{array} \right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -5 & 2 & -3 \\ 0 & 0 & 0 & 5 \end{array} \right]$$

$\text{rank}[A] = 2 \neq \text{rank } [A|B] = 3$

Then the system is inconsistent.